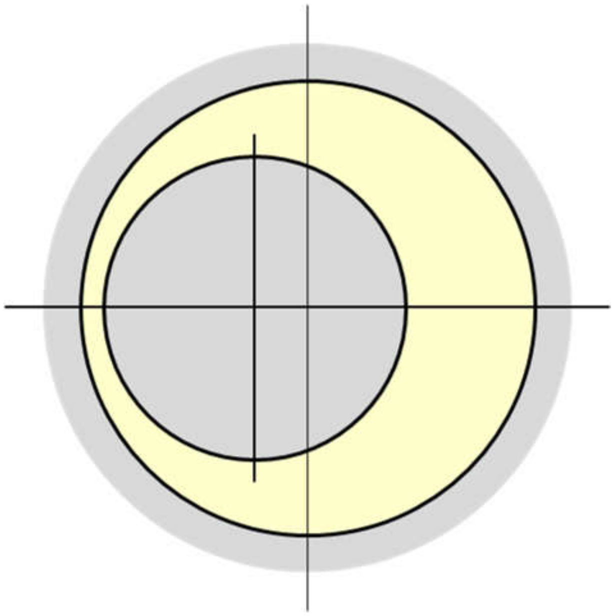

Eccentric Coaxial Cable

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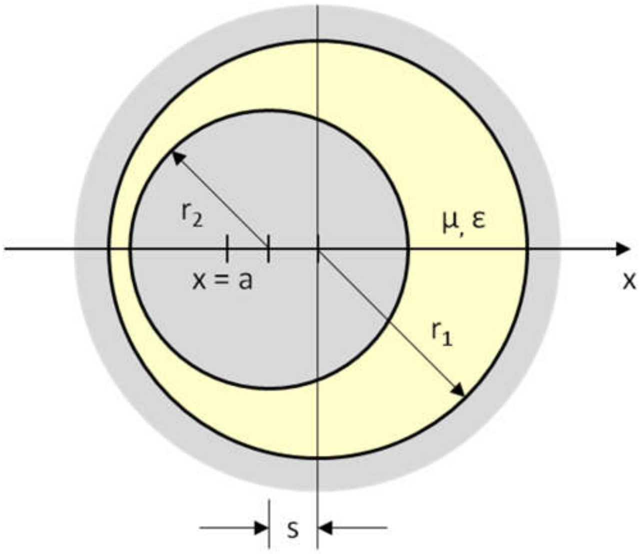
Task description:

Bipolar coordinates are the ideal basis for coaxial cable with an eccentric spaced inner conductor.

The coaxial cable with an outer diameter/ radius r_1 is filled with a non-conducting material with material parameter ε and μ. The inner conductor has the diameter/ radius r_2 and is dislocated by distance s.

The inner conductor has a potential V_0, whereas the outer conductor is grounded (potential free) with V = 0

The so disturbed coaxial cable will have a certain wave resistance, which will be depending on the spacing s.



This problem in one plane could be perfectly described by bipolar cylindrical coordinates.

Task: Calculate the wave resistance of such eccentric coaxial cable and compare the result with the well known solution for a centric conductor of a perfect coaxial cable.

I. Bipolar-Cylindrical Coordinates

I.1 Definition of bipolar-cylindrical coordinates

The following relationship connect a complex w plane and rectangular coordinates z

$$z = a\,\frac{e^{w}+1}{e^{w}-1} = a\,\frac{e^{w/2}+e^{-w/2}}{e^{w/2}-e^{-w/2}} = a\,coth\,\frac{w}{2}$$ equation (I.1.1)

$w = u + jv$ applying the complex w results in

$z = \ a\,coth\,\frac{w}{2} = a\,\frac{sinh\,w}{cosh\,w-1} = a\,\frac{sinh(u+jv)}{cosh(u+jv)-1}$ complex hyperbolic function

$\frac{z}{a} = \frac{\sinh u \cosh jv + \cosh u \sinh jv}{\cosh u \cosh jv + \sinh u \sinh jv - 1}$ based on cosh jv = cos v and sinh jv = j sin v

$\frac{z}{a} = \frac{\sinh u \cos v + j \cosh u \sin v}{\cosh u \cos v + j \sinh u \sin v - 1}\ \frac{\cosh u \cos v - 1 - j \sinh u \sin v}{\cosh u \cos v - 1 - j \sinh u \sin v}$ extended by denominator

$$\frac{z}{a} = \frac{\sinh u \cos v\,(\cosh u \cos v - 1) + \cosh u \sinh u\, sin^2\, v - j\,[sinh^2 u\, sinv \cos v - \cosh u \sin v\,(\cosh u \sin v - 1)]}{(\cosh u \cos v - 1\,)^2 + (\,sinh\,u \sin v\,)^2}$$

The denominator N will be simplified as follows

$N = cosh^2\,u\,cos^2\,v - 2\cosh u \cos v + 1 + sinh^2\,u\,sin^2\,v$ with $\sinh^2 u = \cosh^2 u - 1$

$N = cosh^2\,u\,cos^2\,v - 2\cosh u \cos v + 1 + cosh^2\,u\,sin^2\,v - sin^2\,v$ with $\cos^2 v + \sin^2 v = 1$

$N = cosh^2\,u - 2\cosh u \cos v + cos^2\,v$ will be

$N = (\cosh u - \cos v)^2$

The real part of the numerator Z will be

$Real\,\{Z\} = \sinh u \cosh u\, cos^2\, v - \sinh u \cos v + \sinh u \cosh u\, sin^2\, v = \sinh u\,(\cosh u - \cos v)$

The imagine part of the numerator will be

$Im\,\{Z\} = sinh^2\,u \sin v \cos v - cosh^2\,u \sin v \cos v + \cosh u \sin v = \sin v\,(\cosh u - \cos v\,)$

Therefore exist the following relationship between the z = x + jy plane and w = u + jv plane

$$z = x + j\,y = \frac{a \sinh u}{\cosh u - \cos v} - \,j\,\frac{a \sin v}{\cosh u - \cos v}$$ Equation (I.1.2)

I.1.1 What function describe the value u = constant within the z-plane

$\frac{x}{y} = -\frac{\sinh u}{\sin v}$ squaring

$x^2\,sin^2\,v = y^2\,sinh^2\,u = x^2\,(\,1 - cos^2 v)$

$x = \frac{a \sinh u}{\cosh u - \cos v}$ the real part of eq. I.1.2 will be

$x \cosh u - x \cos v = a \sinh u$ squaring

$x^2\, cos^2\, v = x^2 \cosh^2 u - 2\, a\, x \cosh u \sinh u + a^2\, sinh^2\, u$ inserting

$y^2\, sinh^2\, u \;=\; x^2 - x^2 \cosh^2 u + 2\, a\, x \cosh u \sinh u - a^2\, sinh^2\, u$

$y^2\, sinh^2\, u \;= x^2 - x^2 - x^2\, sinh^2\, u + 2\, a\, x \cosh u \sinh u - a^2\, sinh^2\, u$

$sinh^2\, u\, (\, y^2 + \; a^2 + \; x^2) = \; 2\, a\, x \cosh u \sinh u$

$x^2 + y^2 + a^2 = 2\, a\, x\, \frac{\cosh u}{\sinh u}$

$x^2 - 2\, ax\, \frac{\cosh u}{\sinh u} + \left(a \frac{\cosh u}{\sinh u}\right)^2 \;=\; -a^2 - y^2 + \left(a \frac{\cosh u}{\sinh u}\right)^2$

$\left(x - a \frac{\cosh u}{\sinh u}\right)^2 \;=\; -y^2 + a^2\, \frac{sinh^2\, u}{sinh^2\, u} + a^2\, \frac{cosh^2\, u}{sinh^2\, u}$ which will be at the end

$y^2 + \left(x - \frac{a \cosh u}{\sinh u}\right)^2 \;=\; \frac{a^2}{sinh^2\, u}$ (I.1.3)

This is the equation of a circle with the general form $(x - \; x\,_0)^2 + \; (y - y_0\,)^2 = r^2$

The values u = contant are circles on x-axis with

$(\, x - \; c_u\,)^2 + \; y^2 = (r_u)^2$	is the general circle equation	(I.1.3a)
$c_u = a\, \frac{\cosh u}{\sinh u}$	is the circle centre at x-axis	(I.1.3b)
$r_u = \left\|\frac{a}{\sinh u}\right\|$	is the circle radius	(I.1.3c)

I.1.2 What function describe the value v = constant within the z-plane

As above we will get $\frac{x}{y} = \; - \; \frac{\sinh u}{\sin v}$ squaring

$x^2\, sin^2\, v = y^2\, sinh^2\, u = y^2\, (\, cosh^2 u - 1)$

$y = - \; \frac{a \sin v}{\cosh u - \cos v}$ the imagine part of eq. I.1.2 will be

...

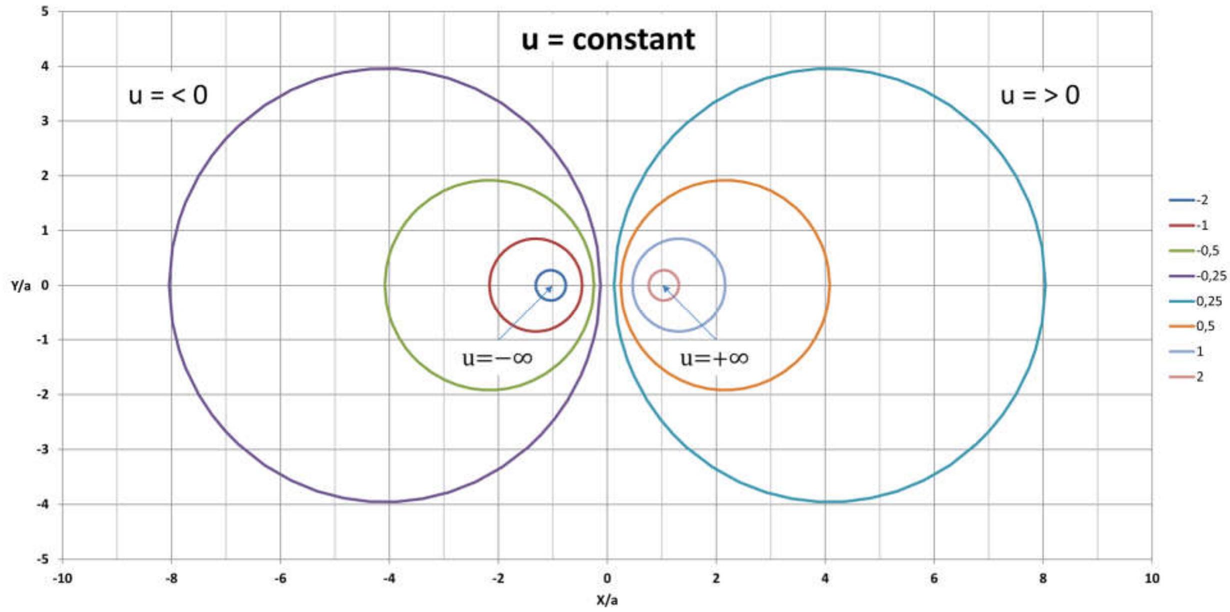
$y^2 + 2\, a\, y\, \frac{\cos v}{\sin v} + \left(a \frac{\cos v}{\sin v}\right)^2 \;=\; a^2 - x^2 + \left(a \frac{\cos v}{\sin v}\right)^2$ will be finally

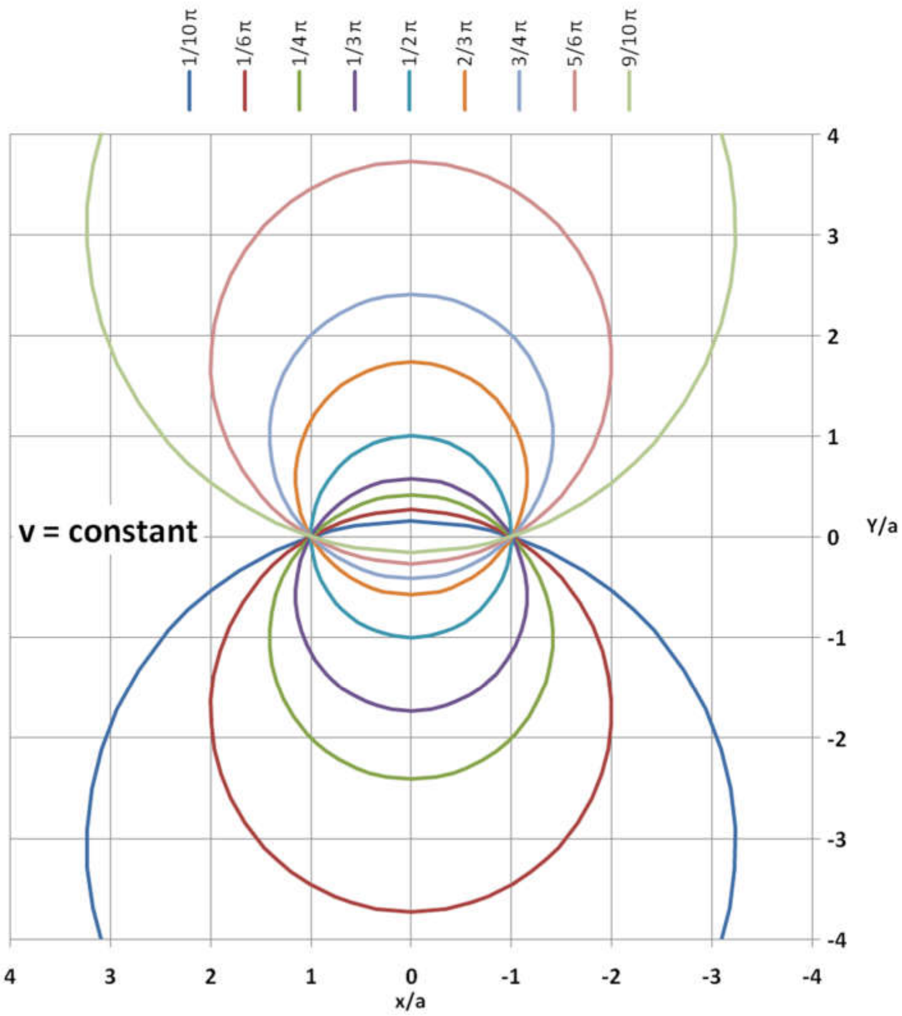
$x^2 + \left(y + a\, \frac{\cos v}{\sin v}\right)^2 = (\frac{a}{\sin v})^2$ (I.1.4)

This is also a circle equation describing with values v = const circles at the y-axis

$x^2 + (\, y - \; c_v\,)^2 = (r_v)^2$	is the general circle equation	(I.1.4a)
$c_v = -\, a\, \frac{\cos v}{\sin v}$	is the circle centre at y-axis	(I.1.4b)
$r_v = \left\|\frac{a}{\sin v}\right\|$	is the circle radius	(I.1.4c)

The following graph demonstrates two set of circles u and v as described in equation I.1.3/ I.1.4.





II Coaxial cable with eccentric inner conductor

II.1 Application of bipolar cylindrical coordinates

The inner and outer conductors will be described by values u = const within the u-v plane

Based on the following equations

$(x - c_u)^2 + y^2 = (r_u)^2$ is the general circle equation see (I.1.3a)

$c_u = a \frac{\cosh u}{\sinh u}$ is the circle centre at x-axis see (I.1.3b)

$r_u = \left|\frac{a}{\sinh u}\right|$ is the circle radius see (I.1.3c)

We will calculate the distance between inner and outer conductor as follows

$$s = c_{u1} - c_{u1} = a \frac{\cosh u_1}{\sinh u_1} - a \frac{\cosh u_2}{\sinh u_2}$$

$$\frac{s}{a} = \frac{\sqrt{1+sinh^2 u_1}}{\sinh u_1} - \frac{\sqrt{1+sinh^2 u_2}}{\sinh u_2} = \frac{\sqrt{1+\left(\frac{a}{r_1}\right)^2}}{\frac{a}{r_1}} - \frac{\sqrt{1+\left(\frac{a}{r_2}\right)^2}}{\frac{a}{r_2}} = \frac{\frac{1}{r_1}\sqrt{{r_1}^2+a^2}}{\frac{a}{r_1}} - \frac{\frac{1}{r_2}\sqrt{{r_2}^2+a^2}}{\frac{a}{r_2}}$$

Therefore the spacing between the circles

$$\boxed{s = \sqrt{{r_1}^2 + a^2} - \sqrt{{r_2}^2 + a^2}}$$

The system constant a will be calculated by substituting $x^2 = {r_1}^2 + a^2$

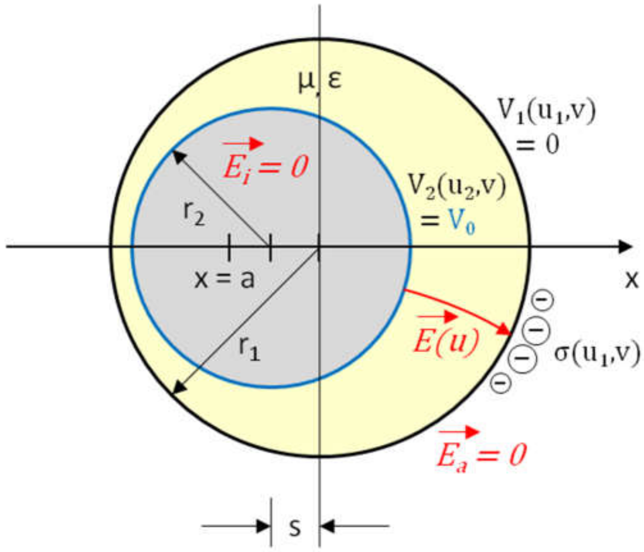
$$s = \sqrt{x^2} - \sqrt{{r_2}^2 + x^2 - {r_1}^2} \quad \gg \quad (s - x)^2 = {r_2}^2 + x^2 - {r_1}^2 = s^2 - 2sx + x^2$$

$$x = \frac{1}{2s}(s^2 + {r_1}^2 - {r_2}^2) = \sqrt{{r_1}^2 + a^2}$$

$$\boxed{a = \sqrt{\left[\frac{1}{2s}(s^2 + {r_1}^2 - {r_2}^2)\right]^2 - {r_1}^2}}$$

system constant (II.1)

II.2 A planar electrostatic potential



The Laplace equation for a potential within a volume without charge density is

$$\Delta V(u, v) = 0$$

Laplace operator in bipolar coordinates

$$\frac{1}{h^2}\left[\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2}\right] = 0$$

Will be reduced to

$$\left[\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2}\right] = 0$$

The general solution is a series in u, v with a combination of sinus and hyperbolic functions.

$$V = (A_0 + B_0\, u)(C_0 + D_0\, v) + \sum_p \sum_q \left[A_p \frac{cosh}{cos}(pu) + B_p \frac{sinh}{sin}(pu)\right]\left[C_p \frac{cos}{cosh}(qv) + D_p \frac{sin}{sinh}(qv)\right]$$

Application to a coaxial cable with boundary condition for the inner and outer conductor

Outer conductor (ground) $V(u_1,v) = 0$

Inner conductor $V(u_2,v) = V_0$

Due to the fact that the potential is independent of v the full series will be simplified as follows

$$V(u_1, v) = A_0 + B_0\, u_1 = 0 \gg \; A_0 = -B_0\, u_1$$

$$V(u_2, v) = A_0 + B_0\, u_2 = V_0 = -B_0\, u_1 + B_0\, u_2 \gg \; B_0 = \frac{V_0}{u_2 - u_1}$$

Therefore the potential in the different regions

$u_2 < u < +\infty$	$V(u) = V_0$	inner centric conductor
$u_2 < u < u_1$	$V(u) = V_0 \frac{u-u_1}{u_2-u_1}$	between inner and outer surface
$u_1 < u < 0$	$V(u) = 0$	outer surface/ ground.

The electrical field will be calculated with as the gradient of V

$$\vec{E} = grad\, V = \frac{1}{h}\left[\vec{e_u}\frac{\partial V}{\partial u} + \vec{e_v}\frac{\partial V}{\partial v}\right] = \vec{e_u}\frac{\cosh u - \cos v}{a}\frac{\partial V}{\partial u}$$

$u_2 < u < +\infty$: $\overrightarrow{E(u)} = 0$

$u_2 < u < u_1$: $\overrightarrow{E(u)} = \vec{e_u}\frac{\cosh u - \cos v}{a}\frac{V_0}{u_2 - u_1}$ (II.2)

$u_1 < u < 0$: $\overrightarrow{E(u)} = 0$

Note: the eccentric coaxial cable is located in the positive x/ u-v plane

II.3 Capacity and wave resistance of eccentric coaxial cable

An electrical field induced at a boundary surface a corresponding surface charge density with

$$\vec{e}\left(\overrightarrow{E_2} - \overrightarrow{E_1}\right) = \frac{\sigma}{\epsilon}$$

For bipolar cylindrical coordinates we will get the following equation

$$\sigma(u_1, v) = \epsilon\, \vec{e}\left(\overrightarrow{E_2(u_1, v)} - \overrightarrow{E_a}\right) \qquad >>> \qquad \sigma(u_1, v) = \epsilon\, \frac{\cosh u - \cos v}{a} \frac{V_0}{u_2 - u_1}$$

The total surface charge resulted out of integration at u_1 for v = − π until + π

$$Q_a = \int_{v=-\pi}^{+\pi} \int \sigma(u_1, v)\, h_v dv\, h_z dz \qquad \text{per length unit in z direction}$$

$$\frac{dQ_q}{dz} = \int_{v=-\pi}^{+\pi} \epsilon\, \frac{\cosh u_1 - \cos v}{a} \frac{V_0}{u_2 - u_1} \frac{a}{\cosh u_1 - \cos v}\, dv$$

Therefore the total charge on the outer boundary

$$\frac{dQ_q}{dz} = 2\pi\varepsilon\, \frac{V_0}{u_2 - u_1} \qquad \text{and with} \qquad Q = C\,V$$

We will get the capacity of a coaxial cable

$$\frac{dC}{dz} = \frac{2\pi\varepsilon}{u_2 - u_1}$$

The inductivity for a TEM line is defined as $L\,C = \mu\varepsilon$

And therefore the inductivity for a coaxial line

$$\frac{dL}{dz} = \frac{\mu}{2\pi}\,(u_2 - u_1) \qquad \text{and with} \qquad Z = \sqrt{\frac{L}{C}}$$

We get finally the wave resistance of an eccentric coaxial cable

$$\boxed{Z_L = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\,(u_2 - u_1)} \qquad \mu = \mu_0\, \mu_r \qquad \varepsilon = \epsilon_0\, \epsilon_r \qquad \sqrt{\frac{\mu_0}{\epsilon_0}} = 376\,\Omega \qquad \text{(II.3)}$$

With the conductor coordinates $u_1 = \sinh^{-1}\frac{a}{r_1}$ $\quad u_2 = \sinh^{-1}\frac{a}{r_2}$

And with the system constant a (see equation II.1) the wave resistance as well as the S11 parameter for reflexion will be calculated as follows:

S parameter $S_{11} = \frac{Z_L - Z_0}{Z_L + Z_0}$ with Z_0 as reference wave resistance for a perfect coax cable

with $$a = \sqrt{\left[\frac{1}{2s}(s^2 + {r_1}^2 - {r_2}^2)\right]^2 - {r_1}^2} = r_1\sqrt{\left[\frac{s^2 + {r_1}^2 - {r_2}^2}{2\,s\,r_1}\right]^2 - 1}$$

therefore $$u_1 = \sinh^{-1}\frac{a}{r_1} = \sinh^{-1}\sqrt{\left[\frac{s^2 + {r_1}^2 - {r_2}^2}{2\,s\,r_1}\right]^2 - 1} = \sinh^{-1} w$$

and $$u_2 = \sinh^{-1}\frac{a}{r_2} = \sinh^{-1}\left(\frac{r_1}{r_2}\sqrt{\left[\frac{s^2 + {r_1}^2 - {r_2}^2}{2\,s\,r_1}\right]^2 - 1}\right) = \sinh^{-1} y = \sinh^{-1}\left(\frac{r_1}{r_2}\, w\right)$$

The u-coordinates $(u_2 - u_1) = \sinh^{-1}\left(\frac{r_1}{r_2} w\right) - \sinh^{-1} w$

And relationship $\sinh^{-1} z = \ln\left(z + \sqrt{1 + z^2}\right)$

For small s we will get a large w, therefore with $\sinh^{-1} z \sim \ln(2z)$

$$(u_2 - u_1) = \sinh^{-1}\left(\frac{r_1}{r_2} w\right) - \sinh^{-1} w \sim \ln\left(2\frac{r_1}{r_2} w\right) - \ln(2w) = \ln\frac{2\frac{r_1}{r_2} w}{2w} = \ln\frac{r_1}{r_2}$$

Therefore the solution for the wave resistance of a perfect coaxial cable

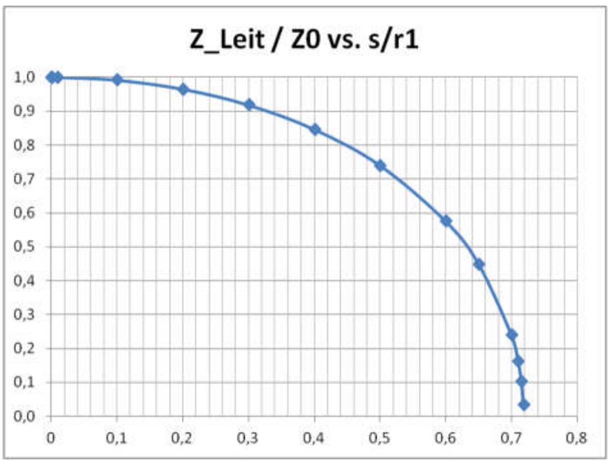
$$Z_L = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \ln\frac{r_1}{r_2}$$

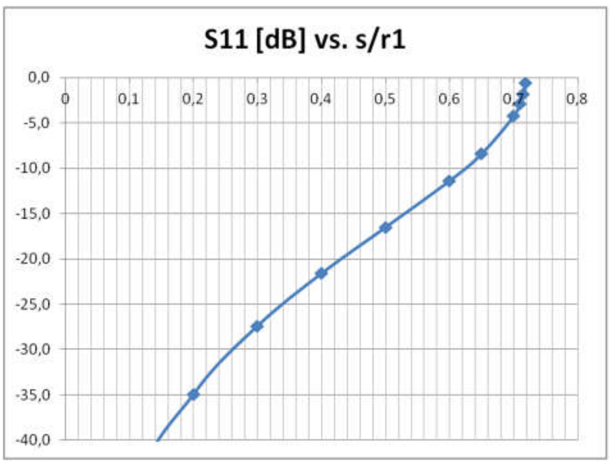
which is in agreement with other calculations of coaxial cable.

II.4 Example: Eccentric coaxial cable

Some values for a typical coax cable:

	Radius_1	1	Radius_2	0,2816348	Eps =	2,3		
					Z0(Coax) =	50,0000		
s/r1	a/r1	u1	u2	u2-u1	Z_Leitung	Z_Leit / Z0	S11	S11 [dB]
0,0001	4603,409	9,127700	10,394844	1,267144	49,999999	1,000000	0,000	-167,36
0,001	460,3403	6,825114	8,092257	1,267143	49,999957	0,999999	0,000	-127,36
0,01	46,02823	4,522520	5,789555	1,267035	49,995714	0,999914	0,000	-87,36
0,1	4,544691	2,218997	3,475210	1,256213	49,568662	0,991373	0,004	-47,27
0,2	2,183617	1,522860	2,745407	1,222547	48,240244	0,964805	0,018	-34,94
0,3	1,355521	1,111855	2,275099	1,163244	45,900232	0,918005	0,043	-27,38
0,4	0,908186	0,814939	1,887203	1,072264	42,310270	0,846205	0,083	-21,59
0,5	0,608684	0,576257	1,513508	0,937251	36,982829	0,739657	0,150	-16,50
0,6	0,372814	0,364677	1,092849	0,728172	28,732790	0,574656	0,270	-11,37
0,65	0,259879	0,257039	0,825682	0,568643	22,437975	0,448759	0,380	-8,39
0,70	0,123766	0,123452	0,426414	0,302962	11,954531	0,239091	0,614	-4,24
0,71	0,082136	0,082044	0,287656	0,205612	8,113210	0,162264	0,721	-2,84
0,715	0,051659	0,051636	0,182412	0,130776	5,160270	0,103205	0,813	-1,80
0,718	0,016931	0,016931	0,060082	0,043151	1,702700	0,034054	0,934	-0,59





Comment: 10% off-centre result in less than -40 dB reflexion, which is an extremely good value.

III. Bibliography/ References

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